Int. J. Solids Structures Vol. 23, No. 8, pp. 1089–1104, 1987
Printed in Great Britain.

0020-7683/87 $3.00+ .00

ON SOME CRACK PROBLEMS FOR INHOMOGENEOUS ELASTIC MATERIALS

W. T. ANG and D. L. CLEMENTS
Department of Applied Mathematics, University of Adelaide, Australia 5001

(*Received* 26 *May* 1986; *in revised form* 12 *September* 1986)

Abstract—The problem of a plane crack in an inhomogeneous material with certain elastic coefficients which exhibit slight variations along the direction perpendicular to the crack is examined in this paper. A series form solution to the problem is proposed and analytical expressions for the first two terms of the series are obtained using a Fourier transform technique. Approximate expressions for the relevant stress intensity factors are also derived.

1. INTRODUCTION

The solution of the problem of a crack in an inhomogeneous material with elastic coefficients which are varying continuously in space presents enormous mathematical difficulties. Hitherto, the problem has been considered only for special cases where the deformation is antiplane or the shear modulus of the material assumes certain specific forms (see, e.g. Clements *et al.*[1] and Dhaliwal and Singh[2]).

In this paper, we examine the problem for an inhomogeneous material which satisfies the conditions of either an antiplane deformation or plane strain. For the case of an antiplane deformation, the shear modulus of the material is assumed to exhibit a slight variation along the direction normal to the crack. For the case of plane strain, Young's modulus of the material varies in a similar fashion, while Poisson's ratio is taken to be a constant. A solution to the problem in series form is assumed. Through the use of a Fourier transform technique, analytical expressions for the first two terms of the series are obtained. Under appropriate conditions, the truncated series obtained by retaining only the first two terms of the series solution provides us with a good approximate solution to the problem. Approximate expressions for the relevant stress intensity factors can then be derived using this solution. Specific cases of the problem (e.g. where the shear elastic modulus shows a linear variation and the stresses act uniformly on the crack) are considered.

2. BASIC EQUATIONS

Neglecting the effect of body forces, the equilibrium equations of an elastic material are given by

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{1}$$

where $i, j = 1, 2, 3$; x_i are the Cartesian coordinates and σ_{ij} are the Cartesian stresses. The usual convention of summing over a repeated Latin suffix is adopted here.

For isotropic materials, the stresses σ_{ij} are related to the strains e_{ij} by

$$\sigma_{ij} = 2\mu e_{ij} + \lambda \delta_{ij} e_{kk} \tag{2}$$

where λ and μ are the Lamé constants (μ is often called the shear modulus), δ_{ij} is the Kronecker delta and the strains e_{ij} are defined as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

where u_i are the Cartesian displacements.

Inverting eqn (2), we obtain

$$e_{ij} = \frac{(1+\nu)}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk} \tag{4}$$

where ν is Poisson's ratio and E is Young's modulus.

For convenience, from now on, we adopt the notations

$$x = x_1, \quad y = x_2, \quad z = x_3, \quad u = u_1, \quad v = u_2, \quad w = u_3,$$

$$[\sigma_{ij}] = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}, \quad \text{and} \quad [e_{ij}] = \begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yz} & e_{yy} & e_{yz} \\ e_{xz} & e_{zy} & e_{zz} \end{pmatrix}.$$

2.1. *Antiplane deformation*

An elastic material is said to undergo an antiplane deformation if $u = 0$, $v = 0$ and w is independent of z. Hence for an antiplane deformation the only non-zero stresses are

$$\sigma_{xz} = \sigma_{zx} = \mu\frac{\partial w}{\partial x}, \qquad \sigma_{yz} = \sigma_{zy} = \mu\frac{\partial w}{\partial y}. \tag{5}$$

From eqn (1) we then have

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) = 0. \tag{6}$$

We assume that the shear modulus μ takes the form

$$\mu = \mu_0 + \varepsilon f(y) \tag{7}$$

where μ_0 is a constant, ε is some constant parameter such that $|\varepsilon| \ll 1$ and f is a given continuous and differentiable function of y. Substituting eqn (7) into eqn (6), we obtain

$$\mu\nabla^2 w + \varepsilon f'(y)\frac{\partial w}{\partial y} = 0 \tag{8}$$

where the prime denotes differentiation with respect to the relevant argument and ∇^2 is the Laplacian operator.

We propose a solution to eqn (8) in the form

$$w = \sum_{n=0}^{\infty} \varepsilon^n \phi_n(x, y). \tag{9}$$

From eqns (5) the stress σ_{yz} is then given by

$$\sigma_{yz} = \sigma_{yz}^{(0)} + \varepsilon\sigma_{yz}^{(1)} + O(\varepsilon^2) \tag{10}$$

where

$$\sigma_{yz}^{(0)} = \mu_0 \frac{\partial \phi_0}{\partial y} \tag{11}$$

and

$$\sigma_{yz}^{(1)} = \mu_0 \frac{\partial \phi_1}{\partial y} + f(y) \frac{\partial \phi_0}{\partial y}. \tag{12}$$

Substituting eqn (9) into eqn (8) and then equating the coefficient of each power of ε to zero, we obtain

$$\nabla^2 \phi_0 = 0 \tag{13}$$

and

$$\nabla^2 \phi_n = -\frac{1}{\mu_0} \left[f(y) \nabla^2 \phi_{n-1} + f'(y) \frac{\partial \phi_{n-1}}{\partial y} \right] \quad \text{for} \quad n > 0. \tag{14}$$

2.2. *Plane strain*

An elastic material satisfies the conditions of plane strain if u and v are independent of z and $w = 0$. From eqns (1)–(3) the equilibrium equations for the elastic material are then

$$\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0, \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0. \end{aligned} \tag{15}$$

We introduce the stress function $\Phi = \Phi(x, y)$ defined in such a way that

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \, \partial y}; \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2}. \tag{16}$$

The stresses as given in eqns (16) satisfy the equilibrium equations (15) exactly.

Now from eqn (3) we have the compatibility of strain condition

$$\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2 \frac{\partial^2 e_{xy}}{\partial x \, \partial y}. \tag{17}$$

Taking Poisson's ratio v to be constant and assuming that Young's modulus E is of the form

$$E = E_0 + \varepsilon h(y) \tag{18}$$

where E_0 is a constant and h is a given continuous and differentiable function of y, through the use of eqn (4) and eqns (16) and (17) we obtain

$$E^3\nabla^2\nabla^2\Phi+[2E(\varepsilon h'(y))^2-E^2\varepsilon h''(y)]\left(\frac{\partial^2\Phi}{\partial y^2}-\frac{\nu}{1-\nu}\frac{\partial^2\Phi}{\partial x^2}\right)$$
$$=2E^2\varepsilon h'(y)\left(\frac{\partial^3\Phi}{\partial x^2\,\partial y}+\frac{\partial^3\Phi}{\partial y^3}\right). \quad (19)$$

If we assume that eqn (19) admits a solution of the form

$$\Phi=\sum_{n=0}^{\infty}\varepsilon^n\Phi_n(x,y) \quad (20)$$

then from eqns (16) we may write

$$\begin{aligned}\sigma_{xx}&=\sigma_{xx}^{(0)}+\varepsilon\sigma_{xx}^{(1)}+O(\varepsilon^2),\\ \sigma_{xy}&=\sigma_{xy}^{(0)}+\varepsilon\sigma_{xy}^{(1)}+O(\varepsilon^2),\\ \sigma_{yy}&=\sigma_{yy}^{(0)}+\varepsilon\sigma_{yy}^{(1)}+O(\varepsilon^2),\end{aligned} \quad (21)$$

where (for $i=0, 1$)

$$\sigma_{xx}^{(i)}=\frac{\partial^2\Phi_i}{\partial y^2},\quad \sigma_{xy}^{(i)}=-\frac{\partial^2\Phi_i}{\partial x\,\partial y},\quad \sigma_{yy}^{(i)}=\frac{\partial^2\Phi_i}{\partial x^2}. \quad (22)$$

Using eqns (4), (16) and (20) and assuming that $|\varepsilon h/E_0|\ll 1$, the displacements u and v can be written as

$$u=u^{(0)}+\varepsilon u^{(1)}+O(\varepsilon^2),\quad v=v^{(0)}+\varepsilon v^{(1)}+O(\varepsilon^2) \quad (23)$$

where $u^{(0)}$, $u^{(1)}$, $v^{(0)}$ and $v^{(1)}$ are given by

$$\begin{aligned}\frac{\partial u^{(0)}}{\partial x}&=\frac{1}{E_0}[(1-\nu^2)\sigma_{xx}^{(0)}-\nu(1+\nu)\sigma_{yy}^{(0)}],\\ \frac{\partial u^{(1)}}{\partial x}&=\frac{1}{E_0}\left[(1-\nu^2)\sigma_{xx}^{(1)}-\nu(1+\nu)\sigma_{yy}^{(1)}-h(y)\frac{\partial u^{(0)}}{\partial x}\right],\\ \frac{\partial v^{(0)}}{\partial y}&=\frac{1}{E_0}[(1-\nu^2)\sigma_{yy}^{(0)}-\nu(1+\nu)\sigma_{xx}^{(0)}],\\ \frac{\partial v^{(1)}}{\partial y}&=\frac{1}{E_0}\left[(1-\nu^2)\sigma_{yy}^{(1)}-\nu(1+\nu)\sigma_{xx}^{(1)}-h(y)\frac{\partial v^{(0)}}{\partial y}\right],\\ \frac{\partial v^{(0)}}{\partial x}+\frac{\partial u^{(0)}}{\partial y}&=\frac{2(1+\nu)}{E_0}\sigma_{xy}^{(0)},\\ \frac{\partial v^{(1)}}{\partial x}+\frac{\partial u^{(1)}}{\partial y}&=\frac{2(1+\nu)}{E_0}\left[\sigma_{xy}^{(1)}-\frac{h(y)}{E_0}\sigma_{xy}^{(0)}\right].\end{aligned} \quad (24)$$

If we are interested in only the first two terms of the series solution (20) then by substituting (20) into (19) we find that it is only necessary to solve

$$\nabla^2\nabla^2\Phi_0=0 \quad (25)$$

and

$$\nabla^2\nabla^2\Phi_1 = \frac{q}{E_0} \tag{26}$$

where

$$q = h''(y)\left[\frac{\partial^2\Phi_0}{\partial y^2} - \frac{\nu}{(1-\nu)}\frac{\partial^2\Phi_0}{\partial x^2}\right] + 2h'(y)\left(\frac{\partial^3\Phi_0}{\partial x^2\,\partial y} + \frac{\partial^3\Phi_0}{\partial y^3}\right). \tag{27}$$

3. AN ANTIPLANE CRACK PROBLEM

3.1. *Statement of the problem*

Consider an infinite elastic material whose shear modulus μ is given by eqn (7) with f being an even function of y. The material contains a crack in the region $|x| < a, y = 0$ (where a is a given positive constant). The material is subject to a small antiplane deformation. An internal stress $\sigma_{yz} = s_0(x)$ (where s_0 in an even function of x) acts on the crack and the displacements and stresses vanish at infinity. The problem is to determine the stress distribution in the neighbourhood of the crack. More specifically, we are interested in calculating the stress intensity factor K defined by

$$K = \lim_{x\to a^+}(x-a)^{1/2}\sigma_{yz}(x,0). \tag{28}$$

From eqn (10) K can be written as

$$K = K^{(0)} + \varepsilon K^{(1)} + O(\varepsilon^2) \tag{29}$$

where

$$K^{(0)} = \lim_{x\to a^+}(x-a)^{1/2}\sigma_{yz}^{(0)}(x,0), \quad K^{(1)} = \lim_{x\to a^+}(x-a)^{1/2}\sigma_{yz}^{(1)}(x,0) \tag{30}$$

where $\sigma_{yz}^{(0)}$ and $\sigma_{yz}^{(1)}$ are defined in eqns (11) and (12), respectively.

From the symmetry about the y-axis, the problem described above is equivalent to the problem of solving eqn (8) subject to the boundary conditions

$$w = 0 \quad \text{for} \quad |x| > a, \quad y = 0 \tag{31}$$

and

$$\sigma_{yz} = s_0(x) \quad \text{for} \quad |x| < a, \quad y = 0. \tag{32}$$

If we make the assumption that this boundary value problem has a solution of the form of eqn (9) and if we are interested in only the first two terms of the series solution (9) then from eqns (9)–(14) and (31) and (32) the problem can be replaced by a set of two problems.

Problem 3.1. Solve eqn (13) subject to

$$\phi_0 = 0 \quad \text{for} \quad |x| > a, \quad y = 0 \tag{33}$$

and

$$\sigma_{yz}^{(0)} = s_0(x) \quad \text{for} \quad |x| < a, \quad y = 0. \tag{34}$$

Problem 3.2. Solve

$$\nabla^2\phi_1 = -\frac{f'(y)}{\mu_0}\frac{\partial\phi_0}{\partial y} \tag{35}$$

subject to

$$\phi_1 = 0 \quad \text{for} \quad |x| > a, \quad y = 0 \tag{36}$$

and

$$\sigma_{yz}^{(1)} = 0 \quad \text{for} \quad |x| < a, \quad y = 0. \tag{37}$$

3.2. *Solution of Problem* 3.1

If we let

$$\phi_0 = \frac{1}{\pi}\int_0^\infty E(\xi)\exp(-\xi y)\cos(\xi x)\,\mathrm{d}\xi \tag{38}$$

where $E(\xi)$ is defined as

$$E(\xi) = \int_0^a r(t)J_0(\xi t)\,\mathrm{d}t \tag{39}$$

where $J_0(x)$ is a Bessel function of order zero and $r(t)$ is a function yet to be determined, it can be readily verified that eqns (13) and (33) are satisfied.

From eqns (11), (38) and (39) and interchanging the order of integration, we obtain

$$\sigma_{yz}^{(0)}(x,0) = -\frac{\mu_0}{\pi}\int_0^a r(t)\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty J_0(\xi t)\sin(\xi x)\,\mathrm{d}\xi\,\mathrm{d}t. \tag{40}$$

Using the results (see Watson[3])

$$\int_0^\infty J_0(\xi t)\sin(\xi x)\,\mathrm{d}\xi = \begin{cases} 0, & \text{for } 0 < x < t \\ (x^2-t^2)^{-1/2}, & \text{for } t < x < \infty \end{cases} \tag{41}$$

it follows that eqn (40) becomes

$$\sigma_{yz}^{(0)}(x,0) = -\frac{\mu_0}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\min(x,a)}\frac{r(t)\,\mathrm{d}t}{(x^2-t^2)^{1/2}}. \tag{42}$$

Hence from eqn (42) condition (34) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{r(t)\,\mathrm{d}t}{(x^2-t^2)^{1/2}} = -\frac{\pi}{\mu_0}s_0(x) \quad \text{for} \quad |x| < a. \tag{43}$$

Equation (43) can be inverted to obtain

$$r(t) = -\frac{2t}{\mu_0}\int_0^t \frac{s_0(u)\,\mathrm{d}u}{(t^2-u^2)^{1/2}}. \tag{44}$$

From eqns (30) and (42) together with integration by parts, the stress intensity factor $K^{(0)}$ can be written as

$$K^{(0)} = \frac{\mu_0}{\pi} \frac{r(a)}{\sqrt{(2a)}}. \tag{45}$$

The value of $r(a)$ can be evaluated either analytically or numerically using eqn (44).

3.3. *Solution of Problem* 3.2

Substituting eqns (38) and (39) together with

$$\phi_1 = \frac{1}{\pi}\int_0^\infty (F(\xi)+G(\xi,y)) \exp(-\xi y) \cos(\xi x)\, \mathrm{d}\xi \tag{46}$$

where $F(\xi)$ and $G(\xi, y)$ are to be determined, into eqn (35) we obtain (after some simplification)

$$\frac{\partial G}{\partial y} - 2\xi G = \frac{\xi}{\mu_0} E(\xi) f(y). \tag{47}$$

The general solution of eqn (47) is

$$G(\xi,y) = \frac{\xi}{\mu_0} E(\xi) \exp(2\xi y)\left[\int^y f(t) \exp(-2\xi t)\, \mathrm{d}t + C\right]$$

where C is an arbitrary function of ξ. Since we require the displacements and stresses to vanish at infinity, the arbitrary function C is set to zero. Thus

$$G(\xi,y) = \frac{\xi}{\mu_0} E(\xi) \exp(2\xi y)\int^y f(t) \exp(-2\xi t)\, \mathrm{d}t. \tag{48}$$

If we choose

$$F(\xi) = \int_0^a v(t) J_0(\xi t)\, \mathrm{d}t - G(\xi, 0) \tag{49}$$

where $v(t)$ is to be determined, condition (36) is satisfied.

Using eqns (12), (41), (42), (46) and (49), we obtain

$$\sigma_{yz}^{(1)}(x,0) = \frac{\mu_0}{\pi}\left[\int_0^\infty \left.\frac{\partial G}{\partial y}\right|_{y=0} \cos(\xi x)\, \mathrm{d}\xi - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\min(x,a)} \frac{v(t)\, \mathrm{d}t}{(x^2-t^2)^{1/2}}\right] - \frac{f(0)}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\min(x,a)} \frac{r(t)\, \mathrm{d}t}{(x^2-t^2)^{1/2}}. \tag{50}$$

Hence from eqns (43) and (50) condition (37) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{v(t)\, \mathrm{d}t}{(x^2-t^2)^{1/2}} = \int_0^\infty \left.\frac{\partial G}{\partial y}\right|_{y=0} \cos(\xi x)\, \mathrm{d}\xi + \frac{\pi f(0)}{\mu_0^2} s_0(x) \quad \text{for} \quad |x| < a. \tag{51}$$

Inverting eqn (51) and using eqn (44), we obtain

$$v(t) = \frac{2t}{\pi}\int_0^t \int_0^\infty \frac{\cos(\xi u)}{(t^2-u^2)^{1/2}} \left.\frac{\partial G}{\partial y}\right|_{y=0} \mathrm{d}\xi\, \mathrm{d}u - \frac{f(0)}{\mu_0} r(t). \tag{52}$$

From eqns (50) and (52), the stress intensity factor $K^{(1)}$ as defined in eqn (30) becomes (after integrating by parts)

$$K^{(1)} = \frac{2a\mu_0}{\pi^2\sqrt{(2a)}}\int_0^a \int_0^\infty \frac{\cos(\xi u)}{(a^2-u^2)^{1/2}} \left.\frac{\partial G}{\partial y}\right|_{y=0} \mathrm{d}\xi\, \mathrm{d}u$$

$$+\frac{\mu_0}{\pi}\lim_{x\to a^+}(x-a)^{1/2}\int_0^\infty \left.\frac{\partial G}{\partial y}\right|_{y=0} \cos(\xi x)\,\mathrm{d}\xi. \tag{53}$$

Provided that the integrals in (53) exist and can be evaluated, the value of $K^{(1)}$ can then be found.

3.4. *Uniform shear*

If we assume that a uniform shear acts on the crack, that is $s_0(x) = -s_0$ (constant), from eqn (44), $r(t)$ is given by

$$r(t) = \frac{s_0\pi t}{\mu_0} \tag{54}$$

and hence from eqn (39) and the results (Watson[3])

$$\int_0^a tJ_0(\xi t)\,\mathrm{d}t = \frac{a}{\xi}J_1(a\xi) \tag{55}$$

$E(\xi)$ becomes

$$E(\xi) = \frac{s_0\pi}{\mu_0}\frac{aJ_1(a\xi)}{\xi} \tag{56}$$

where $J_1(x)$ is a Bessel function of order one. From eqns (45) and (54) the stress intensity factor $K^{(0)}$ is given by

$$K^{(0)} = \frac{s_0 a}{\sqrt{(2a)}}. \tag{57}$$

Consider now the following cases.

Case 3.1 : $f(y) = k|y|$ (*k is a positive constant*). From eqns (48) and (56) we have (for $y \geqslant 0$)

$$G(\xi, y) = -\frac{s_0\pi k}{4\mu_0^2\xi}(2\xi y+1)\frac{aJ_1(a\xi)}{\xi} \tag{58}$$

and hence

$$\frac{\partial G}{\partial y} = -\frac{s_0\pi k}{2\mu_0^2}\frac{aJ_1(a\xi)}{\xi}. \tag{59}$$

Using eqns (53) and (59), the stress intensity factor $K^{(1)}$ is

$$K^{(1)} = -\frac{s_0 a^2 k}{\mu_0 \pi \sqrt{(2a)}}. \tag{60}$$

From eqns (29), (57) and (60) and neglecting $O(\varepsilon^2)$ terms, the stress intensity factor K for this particular case is given by

$$K = \sqrt{\left(\frac{a}{2}\right)} s_0 \left(1 - \frac{\varepsilon k a}{\pi \mu_0}\right). \tag{61}$$

With a proper interpretation of the parameters involved, result (61) may be seen to be consistent with an approximation to a similar stress intensity factor obtained by Clements *et al.*[1]. Result (61) clearly indicates that for this particular type of inhomogeneous material with shear rigidity which increases with $|y|$ the stress intensity factor is smaller in magnitude than the corresponding factor for a material with constant shear modulus μ_0. Furthermore, as the value of μ_0 decreases in magnitude, the difference between these stress intensity factors becomes more pronounced.

Case 3.2: $f(y) = k \exp(-\alpha|y|)$ (*k and α are positive constants*). From eqn (48), we have (for $y \geqslant 0$)

$$G(\xi, y) = -\frac{k\xi}{\mu_0(\alpha+2\xi)} E(\xi) \exp(-\alpha y). \tag{62}$$

Differentiating eqn (62) partially with respect to y, we obtain

$$\frac{\partial G}{\partial y} = \frac{k\alpha\xi}{\mu_0(\alpha+2\xi)} E(\xi) \exp(-\alpha y). \tag{63}$$

Now from eqn (63)

$$\left| \int_0^\infty \frac{\partial G}{\partial y}\bigg|_{y=0} \cos(\xi x)\, d\xi \right| = \frac{k\alpha}{\mu_0} \left| \int_0^\infty \frac{\xi E(\xi)}{(\alpha+2\xi)} \cos(\xi x)\, d\xi \right| \leqslant \frac{k\alpha}{\mu_0} \int_0^\infty |E(\xi)|\, d\xi. \tag{64}$$

The function $E(\xi)$ as given in eqn (56) is finite along the interval $0 \leqslant \xi < \infty$. Its asymptotic behaviour for large ξ then indicates the last integral in eqn (64) is bounded for $x > a$. Hence from eqns (53) and (63) (after interchanging the order of integration)

$$K^{(1)} = \frac{2ak\alpha}{\pi^2 \sqrt{(2a)}} \int_0^\infty \frac{\xi E(\xi)}{(\alpha+2\xi)} \int_0^a \frac{\cos(\xi u)}{(a^2-u^2)^{1/2}}\, du\, d\xi. \tag{65}$$

Using the results (Watson[3])

$$\frac{2}{\pi} \int_0^t \frac{\cos(\xi u)\, du}{(t^2-u^2)^{1/2}} = J_0(\xi t) \tag{66}$$

eqn (65) together with eqn (66) gives

$$K^{(1)} = \frac{s_0 a^2 k\alpha}{\mu_0 \sqrt{(2a)}} \int_0^\infty \frac{J_0(a\xi) J_1(a\xi)\, d\xi}{(\alpha+2\xi)}. \tag{67}$$

The integrand of the integral in eqn (67) behaves as $O(1/\xi^2)$ for large ξ. Hence the infinite integral converges slowly. To speed up its convergence, we rewrite it as

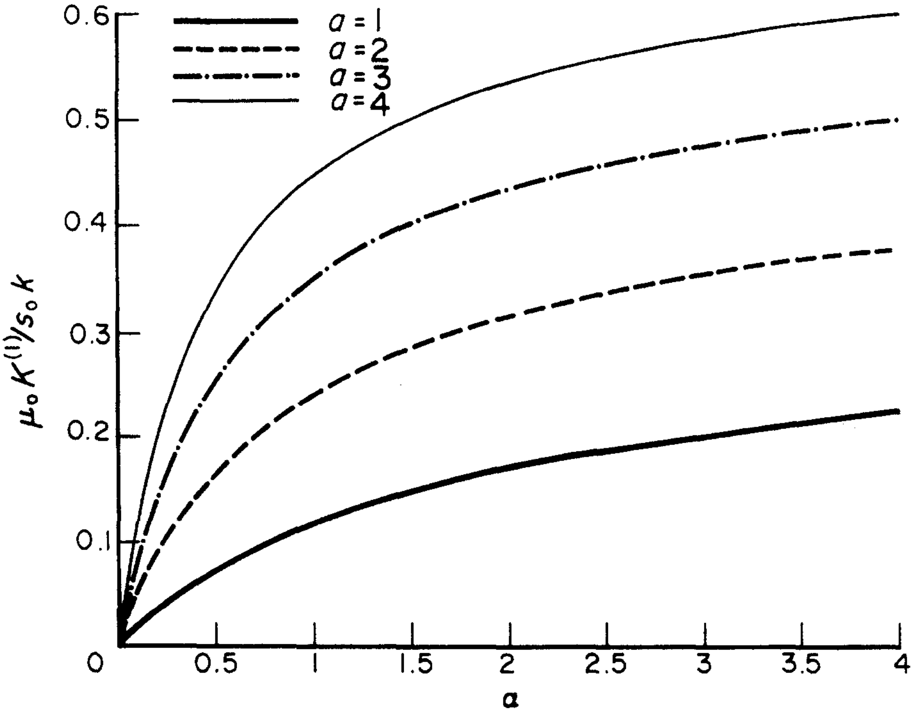

Fig. 1. Antiplane problem: variation of $\mu_0 K^{(1)}/s_0 k$ against α for the case where $\mu = \mu_0 + \varepsilon k \exp(-\alpha|y|)$.

$$\int_0^\infty \frac{J_0(a\xi)J_1(a\xi)}{(\alpha+2\xi)}\,d\xi = \int_0^\infty \frac{1}{\xi}\left[\frac{\xi}{\alpha+2\xi}-\frac{1}{2}\right]J_0(a\xi)J_1(a\xi)\,d\xi + \int_0^\infty \frac{1}{2\xi}J_0(a\xi)J_1(a\xi)\,d\xi. \tag{68}$$

The first integral on the right-hand side of eqn (68) has an integrand which diminishes rapidly as ξ increases and the second integral is given by $1/\pi$ (see Watson[3]). Thus from eqn (68) the stress intensity factor $K^{(1)}$ can now be easily evaluated using an ordinary numerical integration scheme.

In Fig. 1, we plot $\mu_0 K^{(1)}/s_0 k$ against α for various values of a. From the graphs, it is clear that as α increases $K^{(1)}$ increases. Also, for a given value of α, a larger value of a gives rise to $K^{(1)}$ of higher magnitude. The results clearly indicate that for a material with shear modulus $\mu = \mu_0 + \varepsilon k \exp(-\alpha|y|)$ the stress intensity factor K is larger than the corresponding factor for a material with shear modulus μ_0.

4. A PLANE CRACK PROBLEM

4.1. *Statement of the problem*

In this section, we consider the problem of determining the stress distribution in the vicinity of a straight crack in an infinite isotropic material which satisfies the conditions of plane strain. Poisson's ratio ν of the material is assumed to be constant while Young's modulus E varies as in eqn (18) with h being an even function of y. On the crack, which lies in the region $|x| < a, y = 0$, we require the stresses σ_{xy} and σ_{yy} to be such that $\sigma_{xy} = 0$ and $\sigma_{yy} = p_0(x)$ (where p_0 is an even function of x). It is also required that the displacements and stresses vanish at infinity. Of particular interest to us here is the calculation of the stress intensity factor K_I defined by

$$K_I = \lim_{x\to a^+} (x-a)^{1/2}\sigma_{yy}(x,0). \tag{69}$$

From eqn (21), eqn (69) may be rewritten as

$$K_I = K_I^{(0)} + \varepsilon K_I^{(1)} + O(\varepsilon^2) \tag{70}$$

where

$$K_I^{(0)} = \lim_{x \to a^+} (x-a)^{1/2} \sigma_{yy}^{(0)}(x, 0),$$
$$K_I^{(1)} = \lim_{x \to a^+} (x-a)^{1/2} \sigma_{yy}^{(1)}(x, 0). \tag{71}$$

Due to the symmetry of the problem about the y-axis, the problem may be posed as a boundary value problem which involves solving eqn (19) subject to

$$\sigma_{xy} = 0 \quad \text{for all values of } x, \quad y = 0 \tag{72}$$

$$\sigma_{yy} = p_0(x) \quad \text{for} \quad |x| < a, \quad y = 0 \tag{73}$$

and

$$v = 0 \quad \text{for} \quad |x| > a, \quad y = 0. \tag{74}$$

If the first two terms of eqn (20) can provide us with a good approximation to Φ then from eqns (21)–(27) and (72)–(74) this boundary value problem may be replaced by Problems 4.1 and 4.2 below.

Problem 4.1. Solve eqn (25) subject to

$$\sigma_{xy}^{(0)} = 0 \quad \text{for all values of } x, \quad y = 0 \tag{75}$$

$$\sigma_{yy}^{(0)} = p_0(x) \quad \text{for} \quad |x| < a, \quad y = 0 \tag{76}$$

and

$$v^{(0)} = 0 \quad \text{for} \quad |x| > a, \quad y = 0. \tag{77}$$

Problem 4.2. Solve eqn (26) subject to

$$\sigma_{xy}^{(1)} = 0 \quad \text{for all values of } x, \quad y = 0 \tag{78}$$

$$\sigma_{yy}^{(1)} = 0 \quad \text{for} \quad |x| < a, \quad y = 0 \tag{79}$$

and

$$v^{(1)} = 0 \quad \text{for} \quad |x| > a, \quad y = 0. \tag{80}$$

Note that $\sigma_{xy}^{(i)}, \sigma_{yy}^{(i)}$ and $v^{(i)}$ (for $i = 0, 1$) are defined in eqns (22) and (24).

4.2. *Solution of Problem* 4.1

It can be readily verified through direct substitution that eqn (25) admits solution of the form (see Sneddon[4])

$$\Phi_0 = \frac{2}{\pi} \int_0^\infty \frac{\beta(\xi)}{\xi^2} (1 + \xi y) \exp(-\xi y) \cos(\xi x) \, d\xi \tag{81}$$

where $\beta(\xi)$ is yet to be determined.

From eqns (22)–(24) and (81), we obtain

$$\sigma_{yy}^{(0)} = -\frac{2}{\pi}\int_0^\infty \beta(\xi)(1+\xi y)\exp(-\xi y)\cos(\xi x)\,d\xi \tag{82}$$

$$\sigma_{xy}^{(0)} = -\frac{2}{\pi}y\int_0^\infty \xi\beta(\xi)\exp(-\xi y)\sin(\xi x)\,d\xi \tag{83}$$

and

$$v^{(0)} = \frac{2(1+\nu)}{\pi E_0}\int_0^\infty \beta(\xi)(2(1-\nu)+\xi y)\exp(-\xi y)\frac{\cos(\xi x)}{\xi}\,d\xi. \tag{84}$$

Note that the stress $\sigma_{xy}^{(0)}$ as given by eqn (83) satisfies condition (75).

Through the use of eqns (76), (77), (82) and (84) and performing a similar analysis as in Section 3.2, $\beta(\xi)$ is found to be

$$\beta(\xi) = \xi\int_0^a R(t)J_0(\xi t)\,dt \tag{85}$$

where

$$R(t) = -t\int_0^t \frac{p_0(u)\,du}{(t^2-u^2)^{1/2}}. \tag{86}$$

The use of eqns (71), (82), (85) and (86) together with integration by parts yields

$$K_I^{(0)} = \frac{2}{\pi}\frac{R(a)}{\sqrt{(2a)}}. \tag{87}$$

4.3. *Solution of Problem* 4.2

The function Φ_1 defined by

$$\Phi_1(x,y) = \frac{2}{\pi}\int_0^\infty \tilde{G}(\xi,y)\exp(-\xi y)\cos(\xi x)\,d\xi \tag{88}$$

is a solution of eqn (26) if the function $\tilde{G}(\xi,y)$ satisfies

$$\frac{\partial^4\tilde{G}}{\partial y^4} - 4\xi\frac{\partial^3\tilde{G}}{\partial y^3} + 4\xi^2\frac{\partial^2\tilde{G}}{\partial y^2} = \frac{\beta(\xi)}{E_0}\left[\frac{1}{(1-\nu)}h''(y)(y\xi+2\nu-1)+4h'(y)\right]. \tag{89}$$

The general solution of eqn (89) is

$$\tilde{G}(\xi,y) = \tilde{G}_p(\xi,y)+A+By+C\exp(2\xi y)+Dy\exp(2\xi y) \tag{90}$$

where A, B, C and D are arbitrary functions of ξ and $\tilde{G}_p(\xi,y)$ is given by

$$\tilde{G}_p(\xi,y) = -\exp(2\xi y)\int^y W(\xi,t)t\exp(-2\xi t)\,dt + y\exp(2\xi y)\int^y W(\xi,t)\exp(-2\xi t)\,dt \tag{91}$$

where $W(\xi,y)$ is defined by

$$W(\xi, y) = \frac{\beta(\xi)}{E_0(1-\nu)}\left[2\xi(1-2\nu)\int^{y} h(t)\,\mathrm{d}t + (\xi y - 2\nu - 1)h(y)\right]. \tag{92}$$

Since we require the displacements and stresses to vanish at infinity, it is necessary to set the function C and D to zero. The use of condition (78) yields

$$A\xi - B = \left.\frac{\partial \tilde{G}_p}{\partial y}\right|_{y=0} - \xi \tilde{G}_p(\xi, 0). \tag{93}$$

If we assume that the stress $\sigma_{yy}^{(1)}$ is such that

$$\sigma_{yy}^{(1)} = p(x) \quad \text{on} \quad y = 0$$

then from eqn (22) and through the use of a Fourier inversion theorem (in Sneddon[4]) we obtain

$$A = \frac{\gamma(\xi)}{\xi^2} - \tilde{G}_p(\xi, 0) \tag{94}$$

where $\gamma(\xi)$ is defined by

$$\gamma(\xi) = -\int_0^\infty p(u)\cos(\xi u)\,\mathrm{d}u.$$

From eqns (93) and (94), B is given by

$$B = \frac{\gamma(\xi)}{\xi} - \left.\frac{\partial \tilde{G}_p}{\partial y}\right|_{y=0}. \tag{95}$$

To recapitulate, a solution to eqn (26) which satifies condition (78) may be given by

$$\Phi_1 = \frac{2}{\pi}\int_0^\infty \left[\tilde{G}_p(\xi, y) + \frac{\gamma(\xi)}{\xi^2} - \tilde{G}_p(\xi, 0) + \left(\frac{\gamma(\xi)}{\xi} - \left.\frac{\partial \tilde{G}_p}{\partial y}\right|_{y=0}\right)y\right] \times \exp(-\xi y)\cos(\xi x)\,\mathrm{d}\xi. \tag{96}$$

The task now is to determine $\gamma(\xi)$ which satisfies the remaining two boundary conditions of the problem, namely eqns (79) and (80).

Through the use of eqns (22), (24) and (96), we obtain

$$v^{(1)}(x, 0) = \frac{2(1-\nu^2)}{\pi E_0}\int_0^\infty \left[\frac{2\gamma(\xi)}{\xi} + X(\xi) + Z(\xi)\beta(\xi)\right]\cos(\xi x)\,\mathrm{d}\xi \tag{97}$$

$$\sigma_{yy}^{(1)}(x, 0) = -\frac{2}{\pi}\int_0^\infty \gamma(\xi)\cos(\xi x)\,\mathrm{d}\xi \tag{98}$$

where

$$X(\xi) = \frac{1}{\xi^2}\frac{\partial^3 \tilde{G}_p}{\partial y^3}\bigg|_{y=0} - \frac{3}{\xi}\frac{\partial^2 \tilde{G}_p}{\partial y^2}\bigg|_{y=0} \tag{99}$$

$$Z(\xi) = -\frac{2h(0)}{E_0\xi} - \frac{(2\nu-1)h'(0)}{E_0(1-\nu)\xi^2}. \tag{100}$$

If we substitute

$$\gamma(\xi) = -\frac{\xi}{2}\left[X(\xi)+Z(\xi)\beta(\xi)-\int_0^a S(t)J_0(\xi t)\,\mathrm{d}t\right] \tag{101}$$

where $S(t)$ is to be determined, into eqn (97) then condition (80) is satisfied. Together with eqn (98), condition (79) yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{S(t)\,\mathrm{d}t}{(x^2-t^2)^{1/2}} = \Lambda(x) \quad \text{for} \quad |x| < a \tag{102}$$

where

$$\Lambda(x) = \frac{\pi h(0)}{E_0}p_0(x) - \frac{(2\nu-1)h'(0)}{E_0(1-\nu)}\int_x^a \frac{R(t)\,\mathrm{d}t}{(t^2-x^2)^{1/2}} + \int_0^\infty \xi X(\xi)\cos(\xi x)\,\mathrm{d}\xi. \tag{103}$$

Inverting eqn (102), we have

$$S(t) = \frac{2t}{\pi}\int_0^t \frac{\Lambda(u)\,\mathrm{d}u}{(t^2-u^2)^{1/2}}. \tag{104}$$

From eqns (71), (86), (98), (101), (103) and (104) together with integration by parts, we obtain

$$K_I^{(1)} = \frac{2a}{\pi^2\sqrt{(2a)}}\left[\int_0^a\int_0^\infty \frac{\xi X(\xi)}{(a^2-u^2)^{1/2}}\cos(\xi u)\,\mathrm{d}\xi\,\mathrm{d}u - \frac{(2\nu-1)h'(0)}{E_0(1-\nu)}\int_0^a\int_u^a \frac{R(t)\,\mathrm{d}t\,\mathrm{d}u}{(a^2-u^2)^{1/2}(t^2-u^2)^{1/2}}\right] + \frac{1}{\pi}\lim_{x\to a^+}(x-a)^{1/2}\int_0^\infty \xi X(\xi)\cos(\xi x)\,\mathrm{d}\xi. \tag{105}$$

4.4. *Uniform pressure*

If a uniform pressure $p_0(x) = -p_0$ (constant) acts on the crack then eqn (86) gives

$$R(t) = \frac{\pi}{2}p_0 t \tag{106}$$

and from eqn (87) the stress intensity factor $K_I^{(0)}$ is

$$K_I^{(0)} = \frac{p_0 a}{\sqrt{(2a)}}. \tag{107}$$

Substituting eqn (106) into eqn (85) and using eqn (55), we obtain

$$\beta(\xi) = \frac{\pi}{2} p_0 a J_1(a\xi). \tag{108}$$

We now consider the case where Young's modulus E is given by

$$E = E_0 + \varepsilon k|y| \tag{109}$$

where k is a positive constant.

From eqns (91) and (92) and (108) and (109) and differentiating, we obtain

$$\left.\frac{\partial^2 \tilde{G}_p}{\partial y^2}\right|_{y=0} = \frac{k p_0 \pi a J_1(a\xi)}{2E_0\xi}, \qquad \left.\frac{\partial^3 \tilde{G}_p}{\partial y^3}\right|_{y=0} = 0. \tag{110}$$

Using the results (Watson[3])

$$\int_0^\infty \frac{J_1(\xi a)}{\xi} \cos(\xi x)\, d\xi = \begin{cases} (1-x^2/a^2)^{1/2}, & \text{for } 0 < x < a \\ 0, & \text{for } a < x < \infty \end{cases} \tag{111}$$

and from eqns (99), (105), (106) and (110), the stress intensity factor $K_I^{(1)}$ obtained is found to be

$$K_I^{(1)} = -\frac{k p_0 a^2}{\pi E_0 \sqrt{(2a)}}\left(3 + \frac{2\nu-1}{1-\nu}\right). \tag{112}$$

Hence from eqn (69), neglecting $O(\varepsilon^2)$ terms, the stress intensity factor K_I is given by

$$K_I = \sqrt{\left(\frac{a}{2}\right)} p_0 \left[1 - \frac{3\varepsilon k a}{\pi E_0} - \frac{\varepsilon k a}{\pi E_0}\frac{(2\nu-1)}{(1-\nu)}\right]. \tag{113}$$

The stress intensity factor (113) for the plane case indicates behaviour which is qualitatively consistent with the corresponding result (61) and it agrees with that obtained in Rogers and Clements[5] for the case $\nu = 1/2$. That is, the stress intensity factor for a crack in a material with Young's modulus as given in eqn (109) is less than the corresponding factor for a material with Young's modulus E_0. The magnitude of the difference between these two stress intensity factors decreases as E_0 increases. In addition, since for compressible materials Poisson's ratio ν satisfies $0 < \nu < 1/2$, it follows that from eqn (113) the magnitude of the difference between the stress intensity factors for the homogeneous and inhomogeneous materials is bounded below by $2a\sqrt{a} p_0 \varepsilon k/(\sqrt{2}\pi E_0)$ and above by $3a\sqrt{a} p_0 \varepsilon k/(\sqrt{2}\pi E_0)$.

5. SUMMARY

The analysis given in this paper provides us with a means to assess the effect of inhomogeneities on the stress intensity factors for both antiplane and plane crack problems. We assume that the variation of the shear modulus or Young's modulus is slow along the direction perpendicular to the crack. A series form solution to the problem is proposed and the first two terms of the series are obtained by using a Fourier transform technique. As seen in Sections 3.4 and 4.4, for simple variation of this modulus, simple analytical formulae for the first two terms of the stress intensity factors can be obtained when the stresses acting on the crack are constant. If the variation is more complicated, it may still be possible to reduce the expressions for the stress intensity factors to some simpler forms which can then be evaluated numerically (see case 3.2 in Section 3.4). It may be possible to extend the analysis given here to the case where the variation of the modulus is parallel to the crack. Nevertheless, the analysis may become more involved and complicated than the one presented here and it may not be possible to obtain explicit expressions for the stress intensity factors.

REFERENCES

1. D. L. Clements, C. Atkinson and C. Rogers, Antiplane crack problems for an inhomogeneous elastic material. *Acta Mech.* **29**, 199 (1978).
2. R. S. Dhaliwal and B. H. Singh, On the theory of elasticity of a nonhomogeneous medium. *J. Elasticity* **8**, 211 (1978).
3. G. N. Watson, *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge (1922).
4. I. N. Sneddon, *Fourier Transforms*, 1st Edn. McGraw-Hill, New York (1951).
5. C. Rogers and D. L. Clements, Bergman's integral operator method in inhomogeneous elasticity. *Q. Appl. Math.* **36**, 315 (1978).